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An Incremental Carbon Emissions Strategy for the Global Carbon Cycle Using State Variable Feedback Control Design

Keywords: ***Feedback, Control, Stabilization***

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Abstract
Current emissions scenarios are largely based on the inversion of global carbon cycle models. Such inversion analysis relies heavily on the accuracy of the model being inverted. Given that global carbon cycle models are inherently uncertain, it is unclear whether the emissions targets derived from such inversions will achieve the desired stabilization. Here, we explore an alternative strategy in which the definition of the emission scenario is considered as a feedback control problem. It is shown that, provided a reasonable description of the short to medium term dynamics of the global carbon cycle can be identified, this new approach to specifying carbon emissions for policy making appears robust to model uncertainty and exogenous disturbance. It also provides a control law that can be implemented `on-line' and so is adjusted in relation to the latest measured levels of atmospheric CO_2 as time progresses.

1. Introduction
Concerns over the impacts of anthropogenic CO_2 emissions on global climate have focused attention on the need to specify trajectories for emissions that will stabilize atmospheric CO_2 concentrations at acceptable levels. Inversion of global carbon cycle (GCC) models has been the preferred methodology for specifying emissions trajectories at the global scale [1,2,3,4]. By its very nature, inversion is very sensitive to any sources of error or noise. Consequently, there is an essential requirement on the accuracy of the models used to represent the GCC in the inversion computations. If there are inaccuracies in these models, the emissions trajectories, and hence the policy, will not achieve the desired response [5]. This paper considers an alternative strategy in which the definition of the emission scenario is considered as a feedback control problem. This provides an 'on-line' control law that is able to adjust the specified emissions in the light of the measured levels of atmospheric CO_2, hence avoiding over-reliance on the GCC models themselves.

In the current context, the objective of the feedback control design is to specify a controller such that it changes the dynamics of the GCC into those we desire. For the GCC, these criteria would include: the GCC should reach a desired level within an appropriate timeframe; the 'closed loop' system should be stable, well behaved and robust; the constraints on emissions should not be violated; the system should be insensitive to disturbances; and, crucially, the system should be relatively insensitive to modeling inaccuracies. Subsequent sections of this paper show how, by using a model-based control system design strategy that satisfies these criteria, it is possible to attain the WRE 550 atmospheric CO_2 stabilization profile, even in the presence of significant model uncertainty and unexpected exogenous disturbances.

2. Non-Minimal State Space Control System Design
There are many methods of model-based feedback control system design but one of the most successful is the digital Non-Minimal State Space (NMSS) approach of Young *et al*. [6], which is able to mimic many other linear control system design methods (see [7], and the prior references therein) and has been used previously for computing emission strategies [8]. Optimal NMSS control can be achieved in various ways but the most obvious method is to consider the problem in discrete-time (sampled data) terms and select the control input U(k), over the k^{th} year, such that it will minimize the following quadratic cost function,

$$J = \frac{1}{2}\sum_{k=0}^{\infty}(\boldsymbol{X}(k)^T \boldsymbol{Q}\boldsymbol{X}(k) + rU(k)^2) \quad where \quad \boldsymbol{X}(k) = [Y(k)\, Y(k-1) \, ... \, Y(k-n+1)\; U(k-1) \, ... \, U(k-m+1)\; Z(k)]^T ; \quad m > 1 \tag{1}$$
[1]

Here $\boldsymbol{X}(k)$ is the NMSS vector; $Z(k) = Z(k\text{-}1) + \{Y_d(k) - Y(k)\}$ is an 'integral of error state' defined in terms of the error between the system output $Y(k)$ and its desired trajectory $Y_d(k)$ (here the CO_2 stabilization profile); $\boldsymbol{Q}$ is a (n+m) by (n+m), symmetric, positive semi definite matrix; and r is a positive scalar. The selection of

[1] Note that here $m > 1$ is specified since there are no input terms in the NMSS vector when $m = 1$.

Q and r allows for a balance to be achieved between the design objectives and the limitation of control effort required to achieve these objective. The control law derived in this manner takes the form:

$$U(k) = -\boldsymbol{KX}(k) = -K_pY(k) - K_1Y(k-1) - \ldots - K_{n-1}Y(k-n+1) - K_nU(k-1) - \ldots - K_{n+m-2}U(k-m+1) - K_IZ(k) \tag{2}$$

where $\boldsymbol{K}$ is the $n+m$ dimensional control gain vector that optimizes the cost function (1). This optimal linear approach is very general and flexible (see [7]). Moreover, it is also possible to formulate the NMSS design within an optimal, nonlinear, 'model predictive control' setting, where it allows for the imposition of hard constraints on both the input and output variables [9]. The flexibility engendered by these optimal linear and nonlinear formulations of NMSS control system design has an obvious appeal in the present climate control context. If required, for instance, one could include hard constraints, such as rate limitations on $U(k)$, within the control design, reflecting the desire not to derive policies that required unsustainable rates of change of the global socio-economic system. Also, all NMSS designs can be considered and formulated within a multi-objective framework (see [10, 11]), which also has considerable appeal in the present context.

3. Modelling the GCC for Control System Design

In order to evaluate the NMSS approach to control system design in the present context, we first need to identify a model to act as a credible description for the GCC. Considering the system initially in the usual continuous-time terms, a 4th order differential equation model (considered in its equivalent transfer function (TF) form) is used. This model has been estimated directly from the impulse response of the Bern GCC model ([12]; data kindly provided by F. Joos), using the optimal Refined Instrumental Variable for Continuous-time systems (RIVc) algorithm (see e.g. [13]; [14] and the prior references therein), although there is no reason why, if available in an appropriate form, the Bern model itself could not have been used. The estimated parameters are shown in the top part of Table 1.

Fertilization Parameter	**β = 0.011**	**β = 0.046**
	4th Order model	
G_1 (GtC/(GtC/a))	1.4153	1.6759
T_1 (years)	4.9086	4.2265
G_2	11.9776	10.3583
T_2	37.1380	30.6898
G_3	58.1740	35.9790
T_3	277.1496	316.6456
G_4	0.1546	0.1301
T_4	∞	∞
	1st Order model	
$\underline{G}_{SMT}$	36.0634	59.7792
T_{SMT}	49.3800	76.2012

Table 1. Gains (G) and time constants (T) for the partial fraction decomposition of the fourth order continuous time TF representation of the impulse response of the Bern GCC model. These parameterizations explain 99.98% of the variance of Bern model impulse response. G_{SMT} and T_{SMT} are the gains and time constants of the short to medium term response 'control design' model estimated from the data shown in Figure 1 (see text for explanation).

For transparency, the RIVc estimation results in Table 1 are expressed as the sum of first order dynamic components in order to allow for direct comparison with previous studies that have employed the impulse response function (IRF) approach for representing GCC models (e.g. [15, 16]). The two sets of results listed in Table 1 were produced with the CO_2 fertilization parameter, β, in the Bern model set to 0.011 and 0.046, respectively (F. Joos *pers. com.*). The estimated TF parameters resulting from this range of β are later used as the basis for the definition of the model uncertainty in the NMSS control system design. As can be seen from the results in Table 1, the fourth order dynamics of the Bern model impulse response constitute a very 'stiff' dynamic system, comprised of three very different time constants (T_i, i = 1,2,3) with time scales of years, decades, and centuries, together with an integrator that accounts for the mass conservation in the Bern model.

As with all model-based methods, NMSS control system design requires the specification of a 'control design' model for the system to be controlled. Given the ability of the NMSS control system design methodology to handle high order systems, an obvious step would be to use the fourth order TFs in Table 1 directly in the control system design process. However, it has been found that entirely adequate results can be obtained using a simpler, 1st order control design model that adequately captures only the short to medium term (SMT) response in the Bern model data. This is because often, as in this example, the longer-term dynamics can be ignored in control system design since the inherent 'integral-of-error' action in the NMSS controller automatically corrects for any mismatch of this kind in the model. It is not possible to identify a 1st order SMT model directly from the higher order modeling results given in Table 1 because a significant proportion of the SMT response is governed by a process with a century-scale time constant T_3 that has the highest relative gain G_3. An alternative approach is to re-estimate the SMT model based on the response of the GCC to an input that preferentially excites the SMT response relative to the long-term response. There are several such inputs that could be used for this purpose but, conveniently, the near exponential rise in the observed emissions input $U(k)$ over the historic record 1850 – 2000 AD [17], as shown in the upper panel of Figure 1, provides an output CO_2 perturbation $Y(k)$ that proves sufficient for 1st order SMT model parameter estimation. The RIVc algorithm applied to these simulated data yields a first order 'dominant mode' SMT model with gains and time constants G_{SMT} and T_{SMT} that are listed in Table 1. This model explains 99.99% of the variance in simulated output $Y(k)$ shown in Figure 1.

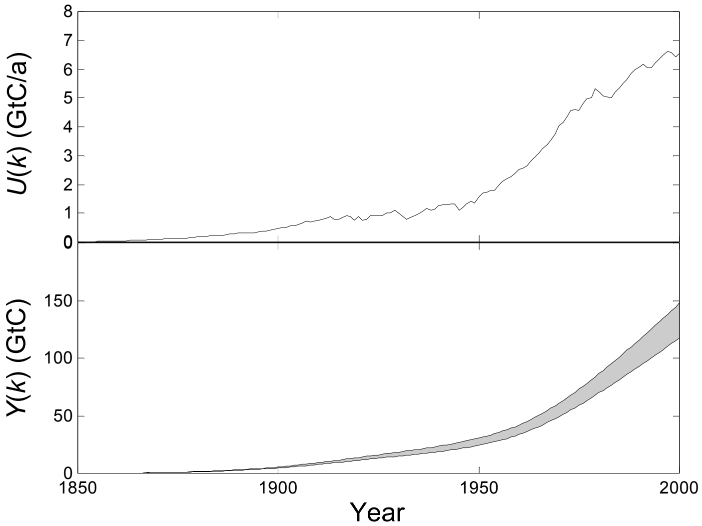

Figure 1. Fossil fuel emissions as estimated by Marland et al [17] (upper panel). The range of simulated response to these emissions (grey shaded) given by the fourth order TF parameters listed in Table 1 (lower panel).

Although a continuous-time control system could be designed using the continuous-time models listed in Table 1, NMSS control is more easily formulated in discrete-time, sampled data terms, as discussed in Section 2. This also makes more practical sense given the discrete nature of observations and mitigation policies. Therefore, it is necessary to specify the equivalent discrete-time SMT model, assuming that the input is constant over the sampling interval Δt (i.e. a 'zero order hold' on the input). It is clear that the selection of Δt for the discrete-time implementation of the control for $Y(k)$ needs to consider the timeframe within which incremental socio-economic change can take place. Here, we will assume Δt = 5 years, whilst recognizing that the choice of Δt is an important issue in its own right. However, the stabilization results are not particularly sensitive to Δt and the NMSS controller could, in principle, produce effective one-step-ahead policy values for $U(k)$ for Δt less than, or considerably larger than, 5 years. For this value of Δt, the discrete-time, difference equation form of the first order SMT model takes the simple form,

$$Y(k) = -aY(k-1) + bU(k-1) \tag{3}$$

where a and b are estimated parameters related to G_{SMT} and T_{SMT}[2].

4. Implementation of NMSS Control: Achieving the CO_2 Stabilization Scenario

The model (3) provides the basis for NMSS control system design. In this case, $\boldsymbol{Q}$ in the cost function (1) is a 2x2 matrix. If this is chosen to be diagonal, with unity elements, and r = 10, then the optimal control gain vector $\boldsymbol{K}$ = [K_p K_i] = [0.0497 0.1856] and [0.0525 0.1884] for the β = 0.011 and β = 0.046 cases respectively, where K_p and K_i are the proportional and integral controls gains respectively. For the purpose of this study, the control law (2) defined in this manner is applied to the fourth order approximation of the Bern model given in Table 1. It is important to highlight the degree of mismatch between the first order SMT model used in the control system design and this fourth order implementation of the system model that is being used to represent the GCC system to be controlled. This means that the control design has been performed with little or no knowledge of the system's long-term response and, in particular, the infinite

[2] $T_{SMT} = -\Delta t/\ln(-a)$ and $G_{SMT} = b/(1+a)$

time response associated with the integration effect of the mass conservation in the Bern model. In order to introduce still more mismatch, the values for ***K*** estimated from the β = 0.011 response are evaluated against the β = 0.046 response and vice versa, in order to represent some of the uncertainty associated with our knowledge of the GCC system. Finally, the ability of the controller to deliver the desired response, despite the difficulty in predicting future events, is evaluated by introducing a significant exogenous disturbance into the simulation of the closed loop response. Here, this disturbance consists of additional inputs of carbon to the atmosphere that are independent of $U(k)$. These could be viewed either as additional anthropogenic emissions, such as land use change, and/or they could represent natural emissions arising from climate change induced carbon sources/sinks. It is felt that the model mismatches, added uncertainty and the exogenous disturbance represent a considerable and yet realistic test for the robustness of the NMSS feedback control law. It is worth noting that, if these additional realistic factors are not present, as in the conventional inversion approach, and the 4th order model is used in the control system design, then the NMSS control law then the NMSS control law could ensure perfect achievement of the stabilization profile but would not then have the robust properties of the current design.

Figure 2 shows the response of the closed loop system when the desired trajectory $Y_d(k)$ is the WRE 550 CO_2 stabilization scenario. It is clear that, irrespective of the uncertainty in the model or the disturbance applied, the controller is able to track the desired WRE 550 profile very well. It is also worth noting that, in the current context, the WRE 550 scenario only arises from the need to specify $Y(k)$ for all k when performing a model inversion. Such a requirement is not essential for the NMSS controller: one can simply specify the desired atmospheric CO_2 at 550 ppmv and then select ***Q*** and r such that the closed loop control yields a stabilization path deemed appropriate. Note that 550 ppmv will always be attained by this control law in the steady state, due to the inherent integral-of-error action in the NMSS controller.

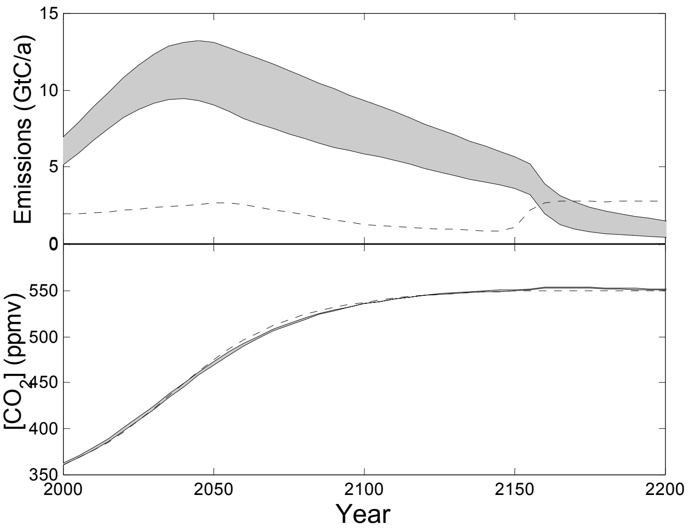

Figure 2. The control emissions envelope (upper panel - grey) arising from the control law (2) forcing a fourth order representation of the Bern GCC model (lower panel - grey) to track the WRE 550 atmospheric CO_2 stabilization scenario (lower panel - dashed). The envelope in emissions and $[CO_2]$ arises from the range of β used to generate the impulse response of the Bern model (see Table 1). The dashed line in the upper panel is an additional exogenous input of carbon into the atmosphere that is acting as a disturbance.

5. Discussion

It should be stressed that Figure 2 does *not* constitute the information on which a policy to stabilize atmospheric CO_2 should be based. Rather, it is a demonstration of the ability of even a simple NMSS control system design to achieve the stabilization scenario objective despite the quite large levels of uncertainty being injected into the simulation. The output in terms of policy is provided by the incremental deployment of the NMSS control law (2), in this case acting on present and past values of the desired and observed response of the GCC, as well as the past value of the control input. Using this on-line control approach, one does not need to know how the GCC will respond in the future; it is sufficient to know only how it has responded up to the present, together with an approximate SMT model of the GCC dynamics that is used to design the control law and ensure closed loop stability. Of course, this approach is not limited to the models used in this exercise; as models of the GCC improve over time, the control law can be refined straightforwardly to reflect any improvement in knowledge. However, even in the absence of any such improvements, the integral-of-error action of the NMSS control will always ensure the measured state of the system $Y(k)$ is driven continually to the desired state $Y_d(k)$ in a smooth trajectory.

It is important to realize that conventional model inversion is essentially an open loop control strategy where observations may be used to parameterize GCC models, but are not fed back to condition the calculation of $U(k)$ thereafter. For example, rapid increases in atmospheric CO_2 such as those recently observed in 2002 and 2003 [18] may require a re-evaluation of emissions strategies derived from inversion studies if they persist. In contrast to this, no re-evaluation of the NMSS control law is required: it will respond appropriately to such measured changes through its continual dependence on the observed response and feedback control action.

The NMSS approach to control system design is very sophisticated (see [7] and the prior references therein) and, in the present paper, it has only been used in its simplest formulation. It is important to emphasize, therefore, that it provides a generic tool that can be applied to higher order, nonlinear, or even multivariable climate models. Moreover, the optimal control setting of the NMSS approach is appealing within a climate management context because, as pointed out in Section 2 of the paper, it allows one to consider nonlinear constrained optimal control, as well as providing a basis for multi-objective optimization.

Finally, one reassuring aspect of our results, regardless of the control system design methodology, is that the observed quasi-exponential profile of historic emissions and atmospheric CO_2 concentrations appears adequate to estimate the SMT model and, hence, to provide the basis for the NMSS feedback control of the real GCC system. This could be extended to consider objectives such as the stabilization of global mean temperature through specifying the appropriate SMT model relating $Y(k)$ to the observed perturbations in global mean temperature.

6. Acknowledgements

We would especially like to thank Professor Fortunat Joos for providing the impulse response data from the Bern GCC model. D.T. Leedal was supported by an NERC studentship number NER/S/A/2002/10392.

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